Self-Duality Condition and Critical Potentials *

C. Cronström [a], Prem P. Srivastava [b], and K. Tanaka [c]

[a] Department of Theoretical Physics, University of Helsinki, 00170 Helsinki, Finland
[b] Centro Brasileiro de Pesquisas Fisicas, Rio de Janeiro, Brasil
[c] Department of Physics, The Ohio State University, Columbus, Ohio 43210, USA

Z. Naturforsch. **52a,** 147–148 (1997)

It is shown that the self-duality constraint on the scalar field (combined with the equations of motion) by itself leads to the critical forms for the potential that minimizes the energy functional in the Chern-Simons-Higgs (CSH) system. If we have only the Chern-Simons (CS) term in the SL (2, R) gauge group one obtains a formalism that yields the equations of motion of a variety on non-linear models in two dimensions when the curvature is set equal to zero.

The abelian Higgs system with both the Maxwell and CS terms in $2+1$ dimensions has drawn much interest recently [1–3]. In the earlier papers, the critical potential leading to self-dual solutions which minimize the energy functional in the CSH system (without the Maxwell term) was obtained [4, 5]. It was also pointed out [6] that the self-duality constraint on the scalar field combined with the equations of motion by itself leads to the critical forms for the potentials in the case when only the CS or only the Maxwell term is present. The notion of self-duality was also extended to the scalar superfield and the critical superpotential was obtained [6].

The Lagrangian for the *bosonic Chern-Simons Higgs* system is ($\hbar = c = 1$)

$$\mathscr{L} = (\tilde{\mathscr{D}}^{\mu} a^*)(\mathscr{D}_{\mu} a) - V(|a|^2) + \frac{\kappa}{4}\varepsilon^{\mu\nu\varrho} v_{\mu} f_{\nu\varrho} - \frac{1}{4} f_{\mu\nu} f^{\mu\nu}, \tag{1}$$

where $\mathscr{D}_{\mu} = \partial_{\mu} - i e v_{\mu}$ and $\tilde{\mathscr{D}}_{\mu} = \partial_{\mu} + i e v_{\mu}$, $\mu = 0, 1, 2$ are the spacetime indices and a is the scalar field.

It was also noted [4, 5] that in the CSH system, without the Maxwell term, the energy functional obeys a Bogomol'nyi-type [7] lower bound when a special choice of the Higgs potential is imposed. The bound is achieved if the scalar field a satisfies the following first order self-duality condition ($i = 1, 2$ and $\varepsilon^{12} = 1$): $\mathscr{D}_1 a = -i\mathscr{D}_2 a$, *or* $\mathscr{D}_i a = -i\varepsilon^{ij}\mathscr{D}_j a$, which may be regarded as the two dimensional analogue of the self-dual gauge field strength in four dimensional space-time.

The general result we find from (1) for the potential using self-duality and the static condition is [8]

$$V'(|a|^2) = e^2 v_0^2 + \frac{e}{\kappa}(2e^2|a|^2 - \partial_i^2) v_0, \tag{2}$$

where v_0 is given by

$$\left[\frac{1}{\kappa^2}(2e^2|a|^2 - \partial_i^2) + 1\right] v_0 = \frac{e}{\kappa}(|a|^2 - C^2). \tag{3}$$

In the limit $\kappa \to 0$, (no CS term), we find

$$V = (e^2/2)(|a|^2 - C^2)^2, \tag{4}$$

and in the limit $e \to \infty$, $\kappa \to \infty$ such that $(e^2/\kappa) \to$ finite, the terms originating from the Maxwell term in the equations of motion drop out. We find

$$V(|a|^2) = (e^2/\kappa)^2(|a|^2 - C^2)^2|a|^2. \tag{5}$$

Both (4) and (5) agree with the previously known results.

Our procedure can be extended [6] also to the *scalar Superfield*

$$\Phi(x, \theta) = a(x) + i\bar{\theta}\psi(x) + i\bar{\theta}\theta f(x). \tag{6}$$

Here $a(x)$ is a complex scalar, $\psi^{\alpha}(x)$ its complex superpartner and $f(x)$ an auxiliary complex scalar. The gauge covariant spinorial derivatives may be defined to be

$$\nabla^{\alpha}\Phi = (D^{\alpha} + e\Gamma^{\alpha})\Phi,$$
$$\tilde{\nabla}^{\alpha}\Phi^* = (D^{\alpha} - e\Gamma^{\alpha})\Phi^*, \tag{7}$$

where α is the spinorial index, D^{α} the covariant spinorial derivative and Γ^{α} a Majorana spinor connection spinor superfield.

* Presented at a Workshop in honor of E. C. G. Sudarshan's contributions to Theoretical Physics, held at the University of Texas in Austin, September 15–17, 1991.

Reprint requests to Prof. Dr. C. Cronström.

0932-0784 / 97 / 0100-0147 $ 06.00 © – Verlag der Zeitschrift für Naturforschung, D-72072 Tübingen

The self-duality constraint on the matter superfield now takes the form [6]

$$\nabla^\alpha \Phi = i(\gamma^0 \nabla)^\alpha \Phi\,, \quad \tilde{\nabla}^\alpha \Phi^* = -i(\gamma^0 \tilde{\nabla})^\alpha \Phi^*\,. \tag{8}$$

The specific (critical) superpotential $V(|\Phi|^2)$ can be obtained and shown to contain the results of the purely bosonic theory without invoking any explicit $N=2$ supersymmetry of the action [9].

Consider next the following *gauged non-abelian* Lagrangian with the CS term [10, 11], $(\hbar = c = e = 1)$,

$$\mathcal{L} = i\psi^\dagger \mathcal{D}_t \psi + \frac{1}{2m}(\mathcal{D}^i\psi)^\dagger(\mathcal{D}_i\psi) + V \tag{9}$$
$$+ \frac{\kappa}{2}\varepsilon^{\mu\nu\varrho}\left(A^a_\mu \partial_\nu A_{\varrho a} + \frac{1}{3} f^a_{bc} A_{\mu a} A^b_\nu A^c_\varrho\right) - \frac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a\,,$$

where ψ is a multiplet of matter fields, $\mu = 0, 1, 2$ or (t, x, y), $A_\mu = A^a_\mu X_a$, X_a being the Lie algebra generators, $\mathcal{D}_\mu = (\partial_\mu + A_\mu)$ and V is the potential to be determined when we impose the static and self-duality conditions. The equations of motion resulting from (9) are

$$i\mathcal{D}_t \psi = -\frac{1}{2m}\mathcal{D}_i\mathcal{D}_i\psi - \frac{\delta V(\psi, \psi^*)}{\delta \psi^*}\,, \tag{10}$$

$$\mathcal{D}_\mu F^{\mu\nu} + \frac{\kappa}{2}\varepsilon^{\nu\varrho\mu}F_{\varrho\mu} = J^\nu\,, \tag{11}$$

where $J^{\mu a} = -\partial\mathcal{L}_{\text{matter}}/\partial A_{\mu a}$.

On adding to them the self-dual equations $\mathcal{D}_i\psi = -i\varepsilon^{ij}\mathcal{D}_j\psi$ and assuming the static configuration, (10) and (11) simplify very much. For example, in the absence of the kinetic term for the gauge field we find, following the procedure described for the abelian case, that A_0 may not vanish, and the critical potential is determined to be [12] $(J^{0a} \equiv \varrho^a)$, $V = (-1/2m\kappa)(\varrho^a \varrho_a) + \textit{const}$, which was assumed at the beginning in [10]. In the presence of the kinetic term, it is possible to choose A_0 to vanish, and we find $V = (-1/4m\kappa)(\varrho^a\varrho_a) + \textit{const}$ while $\kappa = 2m$ is required for consistency.

Finally, we make the following remarks for the case when $\kappa \to \infty$ in (11). We find

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] = 0\,. \tag{12}$$

On the other hand we have, in two dimensions, the curvature two-form [13, 14] $\Omega = d\Gamma + \Gamma \wedge \Gamma$, $\Gamma = \theta_a X_a$, $a = 1, 2, 3$, where θ_a are 1-forms.

Let us consider the X_a to be the generators of SL(2, R), i.e. $X_1 = \left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$, $X_2 = \left(\begin{smallmatrix}0 & 1\\ 0 & 0\end{smallmatrix}\right)$, $X_3 = \left(\begin{smallmatrix}0 & 0\\ 1 & 0\end{smallmatrix}\right)$. Equation (12) is then seen as a zero curvature condition, $\Omega = 0$, if we impose, for example, y independence, viz, $\partial_y(\ ) = 0$ and make the identification $\mu = x$, $\nu = t$, $\Gamma_\mu = A_\mu = A_x$, $\Gamma_\nu = A_\nu = A_t$, and $\Gamma_y = A_y = 0$. Making various choices of A_t and A_x we obtain the non-linear equations in two dimensions like sine-Gordan, modified Korteweg-de Vries (MKdV), non-linear Schrödinger model, KdV, and Liouville equations.

For the case $\partial_t(\ ) = 0$ and the identifications $y = x, \nu = y, \Gamma_t = A = 0$,

$$\Gamma_\mu = A_x = \begin{pmatrix} -\eta & \frac{1}{2}u_x \\ -\frac{1}{2}u_x & \eta \end{pmatrix},$$
$$\Gamma_\nu = A_y = \frac{1}{4\eta}\begin{pmatrix} -\cos u & -\sin u \\ -\sin u & \cos u \end{pmatrix}, \tag{13}$$

where η is a constant, we obtain $u_{xy} - \sin u = 0$, which is the time independent sine-Gordon equation in two dimensions.

Acknowledgements

One of the authors (P. P. S.) acknowledges the hospitality of the Department of Physics at Ohio State University and a fellowship from C.N.Pq of Brasil. The other (K.T.) thanks C. Cronström for the hospitality at the University of Helsinki. This work was supported in part by the U.S. Department of Energy under Contract No. EY-76-C-02-1415*00.

[1] G. V. Dunne and C. A. Trugenberger, Phys. Rev. **D43,** 1323 (1991).
[2] C. Lee, K. Lee, and H. Min, Phys. Letts. **252 B,** 79 (1990).
[3] D. Bazeia, Phys. Rev. **D43,** 4074 (1991).
[4] J. Hong, Y. Kim, and P. Y. Pac, Phys. Rev. Lett. **64,** 2230 (1990).
[5] R. Jackiw and E. Weinberg, Phys. Rev. Lett. **64,** 2234 (1990).
[6] P. P. Srivastava and K. Tanaka, Phys. Letts. **256 B,** 427 (1991).
[7] E. B. Bogomol'nyi, Sov. J. Nucl. Phys. **24,** 449 (1976).
[8] P. P. Srivastava and K. Tanaka, in "Classical and Quantum Systems – Foundations and Symmetries", Proc. of the 2nd International Wigner Symposium, ed. by H. Doebner, W. Scherer, and F. Schroeck, Jr. World Scientific Publ. Co., Singapore 1992.
[9] C. Lee, K. Lee, and E. J. Weinberg, Phys. Letts. **243 B,** 105 (1990).
[10] G. V. Dunne, R. Jackiw, S. Y. Pi, and C. A. Trugenberger, Phys. Rev. **D43,** 1332 (1991).
[11] K. Lee, Phys. Letts. **255 B,** 381 (1991).
[12] P. P. Srivastava and K. Tanaka, (unpublished).
[13] M. Crampin, F. A. E. Pirani, and D. C. Robinson, Lett. Math. Phys. **2,** 15 (1977).
[14] K. Tanaka, J. Math. Phys. **30,** 172 (1989); proceedings, New Theories in Physics, Kazimierz, Poland 1988, editors, Z. Ajduk, S. Pokorski, and A. Trautman, World Scientific, p. 246.